

CURVATURE PROPERTIES OF PSEUDO-SPHERE BUNDLES OVER PARAQUATERNIONIC MANIFOLDS

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ABSTRACT. In this paper we obtain several curvature properties of the twistor and reflector spaces of a paraquaternionic Kähler manifold and prove the existence of both positive and negative mixed 3-Sasakian structures in a principal $SO(2, 1)$ -bundle over a paraquaternionic Kähler manifold.

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1. INTRODUCTION

An almost paraquaternionic structure on a smooth manifold M is a rank 3-subbundle of $\text{End}(TM)$ which is locally spanned by two almost product structures and one almost complex structure which satisfy relations of anti-commutation. Under certain conditions on the holonomy group of a metric adapted to such a structure, one arrives to the concept of paraquaternionic Kähler manifold. If M is a manifold endowed with an almost paraquaternionic structure, then two kinds of unit pseudo-sphere bundle can be considered [1]: the twistor space $Z^-(M)$ of M (the unit pseudo-sphere bundle with fibre the 2-sheeted hyperboloid $S_1^2(-1)$) and the reflector space $Z^+(M)$ of M (the unit pseudo-sphere bundle with fibre the 1-sheeted hyperboloid $S_1^2(1)$). These spaces of great importance in theoretical physics (see e.g. [2, 3]) have been studied recently by several authors [4, 5, 6, 7, 8, 9, 10]. We note that if M is a paraquaternionic Kähler manifold, then the twistor space $Z^-(M)$ admits two canonical almost complex structures I_1 and I_2 (I_1 integrable in dimension greater than 4, I_2 never integrable) and the reflector space $Z^+(M)$ admits two canonical almost para-complex structures P_1 and P_2 (P_1 integrable in dimension greater than 4, P_2 never integrable). Moreover, both $Z^+(M)$ and $Z^-(M)$ carries a natural 1-parameter family of pseudo-Riemannian metrics h_t , $t \neq 0$, such that the canonical projection $\pi_{\pm} : Z^{\pm}(M) \rightarrow M$ is a pseudo-Riemannian submersion [4]. After giving some introductory results, in Section 2, we study the twistor and reflector spaces of a paraquaternionic Kähler manifold M and the almost pseudo-Hermitian geometry of $(Z^-(M), I_i, h_t)$ and $(Z^+(M), P_i, h_t)$, $i = 1, 2$. In Section 3 we give explicit formulas of the pseudo-Riemannian curvature tensors of h_t in terms of the curvature of M and study its properties in details. In particular, we determine the affiliation to Gray and para-Gray classes of $(Z^-(M), I_i, h_t)$ and $(Z^+(M), P_i, h_t)$, $i = 1, 2$.

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On the other hand, the counterpart in odd dimension of a paraquaternionic structure was introduced in [11]. It is called a mixed 3-structure, and appears in a natural way on lightlike hypersurfaces in paraquaternionic Kähler manifolds. Such hypersurfaces inherit two almost paracontact structures and an almost contact structure, satisfying analogous conditions to those satisfied by almost contact 3-structures [12]. This concept has been refined in [13], where the authors have introduced positive and negative metric mixed 3-structures.

W.M. Boothby and H.C. Wang gave in [14] a fibering of a compact manifold M with a regular contact structure as a principal bundle over a symplectic manifold M' with 1-dimensional toroidal group A . Associating a Riemannian metric g' with the symplectic structure, M' is an almost Kählerian manifold and consequently M is a contact metric one. Y. Hatakeyama proved in [15] that there exists a circle bundle over an almost complex manifold that admits a normal contact metric structure. Moreover, K. Sakamoto was proved in [16] that every quaternionic Kähler manifold admits a principal bundle P over it, whose structure group is $SO(3)$, while Konishi [17] constructed a Sasakian 3-structure in P , which is canonically associated with a given quaternion Kähler structure of positive scalar curvature. It can be easily checked that the Sakamoto's construction remains valid if we consider a paraquaternionic Kähler manifold, but now the structure group is $SO(2,1)$. The main purpose of the last Section is to prove that the principal bundle associated with a paraquaternionic Kähler manifold (M, σ, g) admits both a positive and a negative mixed 3-Sasakian structure, provided that M is of non-zero scalar curvature. Moreover, we investigate the existence of non-trivial Einstein metrics in the canonical variation of a pseudo-Riemannian submersion which projects a metric mixed 3-contact structure onto a paraquaternionic Kähler structure.

2. PRELIMINARIES

Let $\tilde{\mathbb{H}}$ be the algebra of paraquaternions and identify $\tilde{\mathbb{H}}^n = \mathbb{R}^{4n}$. It is well known that $\tilde{\mathbb{H}}$ is a real Clifford algebras with $\tilde{\mathbb{H}} = C(1, 1) \cong C(0, 2)$. In fact $\tilde{\mathbb{H}}$ is generated by the unity 1 and generators J_1^0, J_2^0, J_3^0 satisfying the paraquaternionic identities

$$(1) \quad (J_1^0)^2 = (J_2^0)^2 = -(J_3^0)^2 = 1, \quad J_1^0 J_2^0 = -J_2^0 J_1^0 = J_3^0.$$

We assume that $\tilde{\mathbb{H}}$ acts on $\tilde{\mathbb{H}}^n$ by right multiplication and use the convention that $SO(2n, 2n)$ acts on $\tilde{\mathbb{H}}^n$ on the left.

An *almost product structure* on a smooth manifold M is a tensor field P of type $(1,1)$ on M , $P \neq \pm Id$, such that $P^2 = Id$, where Id is the identity tensor field of type $(1,1)$ on M . The pair (M, P) is called an *almost product manifold*. An *almost para-complex manifold* is an almost product manifold (M, P) such that the two eigenbundles T^+M and T^-M associated with the two eigenvalues $+1$ and -1 of P , respectively, have the same rank. Equivalently, a splitting of the tangent bundle TM of a differentiable manifold M , into the Whitney sum of two subbundles TM of the same fiber dimension is called an *almost para-complex structure* on M .

An *almost para-Hermitian structure* on a smooth manifold M is a pair (g, P) , where g is a pseudo-Riemannian metric on M and P is an almost product structure on M , which is compatible with g , i.e. $P^*g = -g$. In this case, the triple (M, g, P) is called an *almost para-Hermitian manifold*. Moreover, (M, g, P) is said to be a *para-Hermitian manifold* if P is integrable, i.e. if the Nijenhuis N_P defined by

$$N_P(X, Y) = [PX, PY] - P[X, PY] - P[PX, Y] + [X, Y]$$

vanishes. An *almost complex structure* on a smooth manifold M is a tensor field J of type $(1, 1)$ on M such that $J^2 = -Id$. The pair (M, J) is called an *almost complex manifold*. We note that the dimension of an almost (para-)complex manifold is necessarily even (see [18, 19]).

An *almost pseudo-Hermitian structure* on a smooth manifold M is a pair (g, J) , where g is a pseudo-Riemannian metric on M and J is an almost complex structure on M , which is compatible with g , i.e. $J^*g = g$. In this case, the triple (M, g, J) is called an *almost pseudo-Hermitian manifold*. Moreover, (M, g, J) is said to be *pseudo-Hermitian* if J is integrable, i.e. if the Nijenhuis N_J defined by

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$$

vanishes.

An *almost para-hypercomplex structure* on a smooth manifold M is a triple $H = (J_1, J_2, J_3)$ of $(1, 1)$ -type tensor fields on M satisfying:

$$(2) \quad J_\alpha^2 = -\tau_\alpha Id, \quad J_\alpha J_\beta = -J_\beta J_\alpha = \tau_\gamma J_\gamma,$$

for any $\alpha \in \{1, 2, 3\}$ and for any even permutation (α, β, γ) of $(1, 2, 3)$, where $\tau_1 = \tau_2 = -1 = -\tau_3$. In this case (M, H) is said to be an *almost para-hypercomplex manifold*. We remark that from (2) it follows that J_1 and J_2 are almost product structures on M , while $J_3 = J_1 J_2$ is an almost complex structure on M . We note that the almost para-hypercomplex structures have been introduced in geometry by P. Libermann [20] under the name of quaternionic structures of second kind (*structures presque quaternioniennes de deuxième espèce*).

A semi-Riemannian metric g on (M, H) is said to be *compatible* or *adapted* to the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=1,2,3}$ if it satisfies:

$$g(J_\alpha X, J_\alpha Y) = \tau_\alpha g(X, Y)$$

for all vector fields X, Y on M and $\alpha \in \{1, 2, 3\}$. Moreover, the pair (g, H) is called an *almost para-hyperhermitian structure* on M and the triple (M, g, H) is said to be an *almost para-hyperhermitian manifold*. We note that any almost para-hyperhermitian manifold is of dimension $4m$, $m \geq 1$, and any adapted metric is necessarily of neutral signature $(2m, 2m)$. If $\{J_1, J_2, J_3\}$ are parallel with respect to the Levi-Civita connection of g , then the manifold is called *para-hyper-Kähler*.

An almost para-hypercomplex manifold (M, H) is called a *para-hypercomplex manifold* if each J_α , $\alpha = 1, 2, 3$, is integrable. In this case H is said to be a *para-hypercomplex structure* on M . Moreover, if g is a semi-Riemannian metric adapted to the para-hypercomplex structure H , then the pair (g, H) is said to be a *para-hyperhermitian structure* on M and (M, g, H) is called a *para-hyperhermitian manifold*.

An *almost paraquaternionic Hermitian manifold* is a triple (M, σ, g) , where M is a smooth manifold, σ is an almost paraquaternionic structure on M , i.e. a rank 3-subbundle of $End(TM)$ which is locally spanned by an almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=1,2,3}$ and g is a compatible metric with respect to H . We remark that, if $\{J_1, J_2, J_3\}$ and $\{J'_1, J'_2, J'_3\}$ are two canonical local bases of σ in U and in another coordinate neighborhood U' of M , then we have for all $x \in U \cap U'$

$$(3) \quad (J'_\alpha)_x = \sum_{\beta=1}^3 s_{\alpha\beta}(x) (J'_\beta)_x, \quad \alpha = 1, 2, 3,$$

where $S_{UU'}(x) = (s_{\alpha\beta}(x))_{\alpha,\beta=1,2,3} \in \text{SO}(2,1)$, because $\{J_1, J_2, J_3\}$ and $\{J'_1, J'_2, J'_3\}$ satisfy the paraquaternionic identities (2).

If (M, σ, g) is an almost paraquaternionic Hermitian manifold such that the bundle σ is preserved by the Levi-Civita connection ∇ of g , then (M, σ, g) is said to be a *paraquaternionic Kähler manifold* [21]. Equivalently, we can write

$$(4) \quad \nabla J_\alpha = \tau_\beta \omega_\gamma \otimes J_\beta - \tau_\gamma \omega_\beta \otimes J_\gamma,$$

where (α, β, γ) is an even permutation of $(1, 2, 3)$ and $\omega_1, \omega_2, \omega_3$ are locally defined 1-forms. We note that the prototype of paraquaternionic Kähler manifold is the paraquaternionic projective space $P^n(\tilde{\mathbb{H}})$ as described by Blažić [22].

If the Riemannian curvature tensor R is taken with the sign convention

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

for all vector fields X, Y, Z on M , then a consequence of (4) is that R satisfies

$$(5) \quad [R, J_\alpha] = \tau_\beta A_\gamma \otimes J_\beta - \tau_\gamma A_\beta \otimes J_\gamma,$$

for any even permutation (α, β, γ) of $(1, 2, 3)$, where

$$A_\alpha = d\omega_\alpha + \tau_\alpha \omega_\beta \wedge \omega_\gamma.$$

If we consider $\Omega_\alpha := g(J_\alpha \cdot, \cdot)$ the fundamental form associated with J_α , $\alpha = 1, 2, 3$, then we have the following structure equations (see [4, 6]):

$$(6) \quad d\omega_\alpha + \tau_\alpha \omega_\beta \wedge \omega_\gamma = \tau_\alpha \nu \Omega_\alpha,$$

for any even permutation (α, β, γ) of $(1, 2, 3)$, where $\nu = \frac{Sc}{4n(n+2)}$ is the reduced scalar curvature, Sc being the scalar curvature defined as the trace of the Ricci tensor ρ .

We recall that the main property of manifolds endowed with this kind of structure is the following.

Theorem 2.1. [21] *Any paraquaternionic Kähler manifold (M, σ, g) is an Einstein space, provided that $\dim M > 4$.*

Let (M, σ, g) be a $4n$ -dimensional paraquaternionic Kähler manifold. Then the Ricci 2-forms of the Levi-Civita connection of g are defined as (see [1]):

$$\rho_\alpha(X, Y) = -\frac{\tau_\alpha}{2} \text{Trace}(Z \rightarrow J_\alpha R(X, Y)Z), \quad \alpha = 1, 2, 3,$$

and for $n > 1$ it follows

$$(7) \quad \rho(X, Y) = \frac{n+2}{n} \rho_\alpha(X, J_\alpha Y), \quad \alpha = 1, 2, 3.$$

Using now Theorem 2.1 we obtain the following relations

$$(8) \quad \rho_\alpha(X, Y) = -\tau_\alpha \frac{Sc}{4(n+2)} g(X, J_\alpha Y), \quad \alpha = 1, 2, 3.$$

On the other hand, from (5) and (7) and taking account of Theorem 2.1, we obtain that the curvature tensor of a $4n$ -dimensional paraquaternionic Kähler manifold ($n > 1$), satisfies (see [21]):

$$\begin{aligned}
 & R(X, Y, J_1 Z, J_1 T) + R(X, Y, Z, T) \\
 &= \frac{Sc}{4n(n+2)} \{g(X, J_3 Y)g(Z, J_3 T) - g(X, J_2 Y)g(Z, J_2 T)\} \\
 & R(X, Y, J_2 Z, J_2 T) + R(X, Y, Z, T) \\
 (9) \quad &= \frac{Sc}{4n(n+2)} \{g(X, J_3 Y)g(Z, J_3 T) - g(X, J_1 Y)g(Z, J_1 T)\} \\
 & R(X, Y, J_3 Z, J_3 T) - R(X, Y, Z, T) \\
 &= \frac{Sc}{4n(n+2)} \{g(X, J_2 Y)g(Z, J_2 T) + g(X, J_1 Y)g(Z, J_1 T)\},
 \end{aligned}$$

for all vector fields X, Y, Z and T on M , where $R(X, Y, Z, T) = g(R(X, Y)Z, T)$.

We consider now the general case of a $4n$ -dimensional smooth manifold M endowed with an almost paraquaternionic structure σ and with a *paraquaternionic connection* ∇ , *i.e.* a linear connection which preserves σ , and following [1] we recall some basic facts concerning the twistor and reflector spaces of M , which will be useful in the next Section.

Let $p \in M$. Any linear frame u on $T_p M$ can be considered as an isomorphism $u : \mathbb{R}^{4n} \rightarrow T_p M$. Taking such a frame u we can define a subspace of the space of the all endomorphisms of $T_p M$ by $u(\mathfrak{sp}(1, \mathbb{R}))u^{-1}$. This subset is a paraquaternionic structure and we define $P(M)$ to be the set of all linear frames u which satisfy $u(\mathfrak{sp}(1, \mathbb{R}))u^{-1} = \sigma$, where $\mathfrak{sp}(1, \mathbb{R}) = \text{Span}\{J_1^0, J_2^0, J_3^0\}$ is the Lie algebra of $\text{Sp}(1, \mathbb{R})$. It is clear that $P(M)$ is the principal frame bundle of M with structure group $\text{GL}(n, \tilde{\mathbb{H}})\text{Sp}(1, \mathbb{R})$, where

$$\text{GL}(n, \tilde{\mathbb{H}}) = \{A \in \text{GL}(4n, \mathbb{R}) | A(\mathfrak{sp}(1, \mathbb{R}))A^{-1} = \mathfrak{sp}(1, \mathbb{R})\}.$$

We denote by $\pi : P(M) \rightarrow M$ the natural projection and remark that the Lie algebra of $\text{GL}(n, \tilde{\mathbb{H}})$ is

$$\mathfrak{gl}(n, \tilde{\mathbb{H}}) = \{A \in \mathfrak{gl}(4n, \mathbb{R}) | AB = BA \text{ for all } B \in \mathfrak{sp}(1, \mathbb{R})\}.$$

We also denote by $(\ , \)$ the inner product in $\mathfrak{gl}(4n, \mathbb{R})$ given by $(A, B) = \text{Trace}(AB^t)$, for $A, B \in \mathfrak{gl}(4n, \mathbb{R})$.

We split now the curvature of ∇ into $\mathfrak{gl}(n, \tilde{\mathbb{H}})$ -valued part R' and $\mathfrak{sp}(1, \mathbb{R})$ -valued part R'' following the classical scheme (see e.g. [23]). We denote the splitting of the $\mathfrak{gl}(n, \tilde{\mathbb{H}}) \oplus \mathfrak{sp}(1, \mathbb{R})$ -valued curvature 2-form Ω on $P(M)$ according to the splitting of the curvature R , by

$$(10) \quad \Omega = \Omega' + \Omega'',$$

where Ω' is a $\mathfrak{gl}(n, \tilde{\mathbb{H}})$ -valued 2-form and Ω'' is a $\mathfrak{sp}(1, \mathbb{R})$ -valued form. Explicitly,

$$(11) \quad \Omega'' = \Omega_1'' J_1^0 + \Omega_2'' J_2^0 + \Omega_3'' J_3^0,$$

where Ω_α'' , $\alpha = 1, 2, 3$ are 2-forms. If $\xi, \eta \in \mathbb{R}^{4n}$, then the 2-forms Ω_α'' , $\alpha = 1, 2, 3$, are given by

$$(12) \quad \Omega_\alpha''(B(\xi), B(\eta)) = \frac{1}{2n} \rho_\alpha(X, Y) = -\tau_\alpha \frac{Sc}{8n(n+2)} g(X, J_\alpha Y),$$

where $X = u(\xi)$, $Y = u(\eta)$ (see [1]).

For each $u \in P(M)$ we consider two linear isomorphism $j^+(u)$ and $j^-(u)$ on $T_{\pi(u)}M$ defined by:

$$j^+(u) = uJ_1^0 u^{-1}, \quad j^-(u) = uJ_3^0 u^{-1}.$$

It is easy to see that

$$(j^-(u))^2 = -Id, \quad (j^+(u))^2 = Id$$

and

$$g(j^-(u)X, j^-(u)Y) = g(X, Y), \quad g(j^+(u)X, j^+(u)Y) = -g(X, Y),$$

for all $X, Y \in T_{\pi(u)}M$.

As in [1], for each $p \in M$ we consider

$$Z_p^\pm(M) = \{j^\pm(u)|u \in P(M), \pi(u) = p\}$$

and we define the *twistor space* Z^- of M and the *reflector space* Z^+ of M , by setting

$$Z^\pm = Z^\pm(M) = \bigcup_{p \in M} Z_p^\pm(M).$$

Then the twistor space $Z^-(M)$ is the unit pseudo-sphere bundle with fibre the 2-sheeted hyperboloid $S_1^2(-1) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = -1\}$ and the reflector space $Z^+(M)$ is the unit pseudo-sphere bundle with fibre the 1-sheeted hyperboloid $S_1^2(1) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = 1\}$.

We denote by A^* (resp. $B(\xi)$) the fundamental vector field (resp. the standard horizontal vector field) on $P(M)$, the principal frame bundle of M , corresponding to $A \in \mathfrak{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathfrak{sp}(1, \mathbb{R})$ (resp. $\xi \in \mathbb{R}^{4n}$). Let $u \in P(M)$ and Q_u be the horizontal subspace of the tangent space $T_u P(M)$ induced by the connection ∇ on M (see [19]). As in [1] we have the decompositions

$$T_u P(M) = (h_i)_u^* \oplus (m_i)_u^* \oplus Q_u, \quad i = 1, 3$$

and the following isomorphisms

$$\begin{aligned} j_{*u}^-|_{(m_3)_u^* \oplus Q_u} : (m_3)_u^* \oplus Q_u &\rightarrow T_{j^-(u)} Z^-, \\ j_{*u}^+|_{(m_1)_u^* \oplus Q_u} : (m_1)_u^* \oplus Q_u &\rightarrow T_{j^+(u)} Z^+ \end{aligned}$$

where

$$\begin{aligned} h_3 &= \{A \in \mathfrak{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathfrak{sp}(1, \mathbb{R}) | AJ_3^0 = J_3^0 A\}, \\ h_1 &= \{A \in \mathfrak{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathfrak{sp}(1, \mathbb{R}) | AJ_1^0 = J_1^0 A\}, \end{aligned}$$

$$(13) \quad m_3 = \{A \in \mathfrak{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathfrak{sp}(1, \mathbb{R}) | AJ_3^0 = -J_3^0 A\} = \text{Span}\{J_1^0, J_2^0\},$$

$$(14) \quad m_1 = \{A \in \mathfrak{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathfrak{sp}(1, \mathbb{R}) | AJ_1^0 = -J_1^0 A\} = \text{Span}\{J_2^0, J_3^0\},$$

and

$$(h_i)_u^* = \{A_u^* | A \in h_i\}, \quad (m_i)_u^* = \{A_u^* | A \in m_i\}, \quad i = 1, 3.$$

Now, we can define two almost complex structures I_1 and I_2 on Z^- by (see [1])

$$\begin{aligned} (15) \quad I_1(j_{*u}^- A^*) &= j_{*u}^-(J_3^0 A)^*, \\ I_2(j_{*u}^- A^*) &= -j_{*u}^-(J_3^0 A)^* \\ I_i(j_{*u}^- B(\xi)) &= j_{*u}^- B(J_3^0 \xi), \quad i = 1, 2, \end{aligned}$$

for $u \in P(M)$, $A \in m_3$, $\xi \in \mathbb{R}^{4n}$.

Similarly, it can be defined two almost paracomplex structures P_1 and P_2 on Z^+ by (see [1])

$$(16) \quad \begin{aligned} P_1(j_{*u}^+ A^*) &= j_{*u}^+(J_1^0 A)^*, \\ P_2(j_{*u}^+ A^*) &= -j_{*u}^+(J_1^0 A)^* \\ P_i(j_{*u}^+ B(\xi)) &= j_{*u}^+ B(J_1^0 \xi), \quad i = 1, 2, \end{aligned}$$

for $u \in P(M)$, $A \in m_1$, $\xi \in \mathbb{R}^{4n}$.

We remark that the almost complex structures defined above were also defined and investigated in [6] on paraquaternionic Kähler manifolds, the authors proving that the almost complex structure I_2 is never integrable while I_1 is always integrable. Moreover, we note that in [1] the authors found that the para-complex structures P_2 is never integrable on reflector space, while P_1 is always integrable.

3. CURVATURE PROPERTIES OF (Z^-, I_i, h_t) AND (Z^+, P_i, h_t) , $i = 1, 2$ ON PARAQUATERNIONIC KÄHLER MANIFOLDS

Let (M, σ, g) be a $4n$ -dimensional paraquaternionic Kähler manifold. Since the Levi-Civita connection ∇ of g is a paraquaternionic connection on M , it follows that all the considerations in the last part of the above sections remain valid if the manifold is endowed with a paraquaternionic Kähler structure. Then, as in [4, 6, 9], we can define a natural 1-parameter family of pseudo-Riemannian metrics h_t , $t \neq 0$, on Z^- by

$$(17) \quad \begin{aligned} h_t(j_{*u}^- A^*, j_{*u}^- B^*) &= t(A, B), \\ h_t(j_{*u}^- A^*, j_{*u}^- B(\xi)) &= 0, \\ h_t(j_{*u}^- B(\xi), j_{*u}^- B(\eta)) &= g(X, Y), \end{aligned}$$

for $u \in P(M)$, $A, B \in m_3$, $\xi, \eta \in \mathbb{R}^{4n}$ and $X = u(\xi)$, $Y = u(\eta)$, and similarly on Z^+ :

$$(18) \quad \begin{aligned} h_t(j_{*u}^+ A^*, j_{*u}^+ B^*) &= t(A, B), \\ h_t(j_{*u}^+ A^*, j_{*u}^+ B(\xi)) &= 0, \\ h_t(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta)) &= g(X, Y), \end{aligned}$$

for $u \in P(M)$, $A, B \in m_1$, $\xi, \eta \in \mathbb{R}^{4n}$ and $X = u(\xi)$, $Y = u(\eta)$.

It is easy now to see that (I_i, h_t) , $i = 1, 2$ (resp. (P_i, h_t) , $i = 1, 2$) determine two families of almost pseudo-Hermitian structures on Z^- (resp. two families of almost para-Hermitian structures on Z^+).

The computations performed in [24] for the metric twistor space over Riemannian manifold M can be adapted in our cases, using (15) and (17) for the twistor space and respectively (16) and (18) for the reflector space. We obtain for the curvature tensors K^\pm , where K^- is the curvature tensor on the twistor space and K^+ is the

curvature tensor on the reflector space, the following expressions:

$$\begin{aligned}
(19) \quad & K^\pm(j_{*u}^\pm A^*, j_{*u}^\pm B^*, j_{*u}^\pm C^*, j_{*u}^\pm D^*) = -t([A, B], [C, D]) \\
& K^\pm(j_{*u}^\pm A^*, j_{*u}^\pm B^*, j_{*u}^\pm C^*, j_{*u}^\pm B(\xi)) = 0 \\
& K^\pm(j_{*u}^\pm A^*, j_{*u}^\pm B(\xi), j_{*u}^\pm B^*, j_{*u}^\pm B(\eta)) = \frac{t}{2}([A, B], \Omega(B(\xi), B(\eta))(u)) - \\
& \quad - \frac{t^2}{4}\varepsilon_i(B, \Omega(B(\xi), B(e_i))(u))(A, \Omega(B(\eta), B(e_i))(u)) \\
& K^\pm(j_{*u}^\pm B(\xi), j_{*u}^\pm B(\eta), j_{*u}^\pm B(\zeta), j_{*u}^\pm A^*) = \frac{t}{2}(A, B(\zeta)_u(\Omega(B(\xi), B(\eta)))) \\
& K^\pm(j_{*u}^\pm B(\xi), j_{*u}^\pm B(\eta), j_{*u}^\pm B(\zeta), j_{*u}^\pm B(\tau)) = R(u(\xi), u(\eta), u(\zeta), u(\tau)) - \\
& \quad - \frac{t}{4}\{(\Omega_{m_\pm}(B(\eta), B(\zeta))(u), \Omega_{m_\pm}(B(\xi), B(\tau))(u)) - \\
& \quad - (\Omega_{m_\pm}(B(\xi), B(\zeta))(u), \Omega_{m_\pm}(B(\eta), B(\tau))(u)) - \\
& \quad - 2(\Omega_{m_\pm}(B(\xi), B(\eta))(u), \Omega_{m_\pm}(B(\zeta), B(\tau))(u))\}
\end{aligned}$$

for $u \in P(M)$, $A, B, C, D \in m_\pm$, where $m_- = m_3$, $m_+ = m_1$, $\xi, \eta, \zeta, \tau \in \mathbb{R}^{4n}$ and Ω_{m_\pm} denotes the m_\pm -component of Ω .

Using now (10)–(14) in (19) we derive

$$\begin{aligned}
(20) \quad & K^-(j_{*u}^- A^*, j_{*u}^- B^*, j_{*u}^- C^*, j_{*u}^- D^*) = \frac{t}{n}(J_3^0 A, B)(J_3^0 C, D) \\
& K^-(j_{*u}^- A^*, j_{*u}^- B^*, j_{*u}^- C^*, j_{*u}^- B(\xi)) = 0 \\
& K^-(j_{*u}^- A^*, j_{*u}^- B(\xi), j_{*u}^- B^*, j_{*u}^- B(\eta)) = \\
& = (-\frac{t^2 Sc^2}{64n(n+2)^2} + \frac{tSc}{8n(n+2)})(J_3^0 A, B)g(J_3 X, Y) + \frac{t^2 Sc^2}{64n(n+2)^2}(A, B)g(X, Y) \\
& K^-(j_{*u}^- B(\xi), j_{*u}^- B(\eta), j_{*u}^- B(\zeta), j_{*u}^- A^*) = 0 \\
& K^-(j_{*u}^- B(\xi), j_{*u}^- B(\eta), j_{*u}^- B(\zeta), j_{*u}^- B(\tau)) = R(X, Y, Z, T) - \\
& \quad - \frac{tSc^2}{64n(n+2)^2}\{g(Y, J_1 Z)g(X, J_1 T) + g(Y, J_2 Z)g(X, J_2 T) - \\
& \quad - g(X, J_1 Z)g(Y, J_1 T) - g(X, J_2 Z)g(Y, J_2 T) - \\
& \quad - 2g(X, J_1 Y)g(Z, J_1 T) - 2g(X, J_2 Y)g(Z, J_2 T)\}
\end{aligned}$$

for $u \in P(M)$, $A, B, C, D \in m_3$, $\xi, \eta, \zeta, \tau \in \mathbb{R}^{4n}$ and $u(\xi) = X$, $u(\eta) = Y$, $u(\zeta) = Z$, $u(\tau) = T$ and

$$\begin{aligned}
(21) \quad & K^+(j_{*u}^+ A^*, j_{*u}^+ B^*, j_{*u}^+ C^*, j_{*u}^+ D^*) = \frac{t}{n}(J_1^0 A, B)(J_1^0 C, D) \\
& K^+(j_{*u}^+ A^*, j_{*u}^+ B^*, j_{*u}^+ C^*, j_{*u}^+ B(\xi)) = 0 \\
& K^+(j_{*u}^+ A^*, j_{*u}^+ B(\xi), j_{*u}^+ B^*, j_{*u}^+ B(\eta)) = \\
& = (-\frac{t^2 Sc^2}{64n(n+2)^2} - \frac{tSc}{8n(n+2)})(J_1^0 A, B)g(J_1 X, Y) - \frac{t^2 Sc^2}{64n(n+2)^2}(A, B)g(X, Y) \\
& K^+(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta), j_{*u}^+ B(\zeta), j_{*u}^+ A^*) = 0 \\
& K^+(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta), j_{*u}^+ B(\zeta), j_{*u}^+ B(\tau)) = R(X, Y, Z, T) - \\
& \quad - \frac{tSc^2}{64n(n+2)^2}\{-g(Y, J_2 Z)g(X, J_2 T) + g(Y, J_3 Z)g(X, J_3 T) + \\
& \quad + g(X, J_2 Z)g(Y, J_2 T) - g(X, J_3 Z)g(Y, J_3 T) + \\
& \quad + 2g(X, J_2 Y)g(Z, J_2 T) - 2g(X, J_3 Y)g(Z, J_3 T)\}
\end{aligned}$$

for $u \in P(M)$, $A, B, C, D \in m_1$, $\xi, \eta, \zeta, \tau \in \mathbb{R}^{4n}$ and $u(\xi) = X$, $u(\eta) = Y$, $u(\zeta) = Z$, $u(\tau) = T$.

We recall now that the $*$ -Ricci tensor of a $2n$ -dimensional almost pseudo-Hermitian manifold (M, g, J) is defined by

$$\rho^*(X, Y) = \sum_{i=1}^{2n} \varepsilon_i R(X, E_i, JY, JE_i),$$

where R denotes the curvature of the metric g , $\{E_1, \dots, E_{2n}\}$ is a pseudo-orthonormal basis at an arbitrary point p and X, Y are tangent vectors at p . If the $*$ -Ricci tensor is scalar multiple of the metric then the manifold is said to be $*$ -Einstein.

On the other hand, we note that A. Gray introduced in [25] three basic classes $AH1$, $AH2$, $AH3$ of almost Hermitian manifolds, whose curvature tensors resemble that of a Kähler manifold. They are defined by the following curvature identities:

$$AH1 : R(X, Y, Z, T) = R(X, Y, JZ, JT),$$

$$AH2 : R(X, Y, Z, T) = R(JX, JY, Z, T) + R(JX, Y, JZ, T) + R(JX, Y, Z, JT),$$

$$AH3 : R(X, Y, Z, T) = R(JX, JY, JZ, JT),$$

where R is the curvature tensor of the manifold. It is easy to see that

$$AH1 \subset AH2 \subset AH3.$$

By analogy, one says that an almost para-Hermitian manifold satisfies the para-Gray identities if

$$APH1 : R(X, Y, Z, T) = -R(X, Y, JZ, JT),$$

$$APH2 : R(X, Y, Z, T) = -R(JX, JY, Z, T) - R(JX, Y, JZ, T) - R(JX, Y, Z, JT),$$

$$APH3 : R(X, Y, Z, T) = R(JX, JY, JZ, JT),$$

where R is the curvature tensor of the manifold. We note that para-Gray-like identities were considered in [26, 27]) and it can be easily checked that

$$APH1 \subset APH2 \subset APH3.$$

We can give now the main curvature properties of (Z^-, I_i, h_t) and (Z^+, P_i, h_t) , $i = 1, 2$, in the following two theorems.

Theorem 3.1. *Let (M, σ, g) be a $4n$ -dimensional paraquaternionic Kähler manifold and (Z^-, I_i, h_t) , $i = 1, 2$ the twistor spaces associated. Then:*

- (i) *The manifolds (Z^-, I_i, h_t) , $i = 1, 2$, belong always to $AH2$ and $AH3$ and are with pseudo-Hermitian Ricci tensor and with pseudo-Hermitian $*$ -Ricci tensor;*
- (ii) *The manifolds (Z^-, I_1, h_t) belong to $AH1$ iff $Sc = 0$ or $Sc = \frac{4(n+2)}{t}$;*
- (iii) *The manifolds (Z^-, I_2, h_t) belong to $AH1$ iff $Sc = 0$;*
- (iv) *The manifolds (Z^-, I_i, h_t) , $i = 1, 2$ are Einstein iff*

$$Sc = \frac{4(n+2)}{t} \text{ or } Sc = \frac{4(n+2)}{(n+1)t};$$

- (v) *The manifolds (Z^-, I_1, h_t) are $*$ -Einstein iff*

$$Sc = \frac{4(n+2)}{t} \text{ or } Sc = -\frac{4(n+2)}{nt};$$

- (vi) *The manifolds (Z^-, I_2, h_t) are $*$ -Einstein iff*

$$Sc = \frac{2(n+2)}{(n-1)t} [3n - 1 - \sqrt{9n^2 - 10n + 5}]$$

or

$$Sc = \frac{2(n+2)}{(n-1)t} [3n-1 + \sqrt{9n^2 - 10n + 5}].$$

Proof. (i) This statement follows from (1), (9), (13), (15) and (20) after some long but straightforward computations.

(ii) + (iii) From (15) and (20) we obtain

$$\begin{aligned}
 & K^-(j_{*u}^- A^*, j_{*u}^- B^*, I_i(j_{*u}^- C^*), I_i(j_{*u}^- D^*)) = \frac{1}{n}(J_3^0 A, B)(J_3^0 C, D) \\
 & K^-(j_{*u}^- A^*, j_{*u}^- B^*, I_i(j_{*u}^- C^*), I_i(j_{*u}^- B(\xi))) = 0 \\
 & K^-(j_{*u}^- A^*, j_{*u}^- B(\xi), I_i(j_{*u}^- B^*), I_i(j_{*u}^- B(\eta))) = \\
 (22) \quad & = (\varepsilon_i \frac{t^2 Sc^2}{64n(n+2)^2} - \varepsilon_i \frac{tSc}{8n(n+2)})(A, B)g(X, Y) - \varepsilon_i \frac{t^2 Sc^2}{64n(n+2)^2} (J_3^0 A, B)g(J_3 X, Y) \\
 & K^-(j_{*u}^- B(\xi), j_{*u}^- B(\eta), I_i(j_{*u}^- B(\zeta)), I_i(j_{*u}^- A^*)) = 0 \\
 & K^-(j_{*u}^- B(\xi), j_{*u}^- B(\eta), I_i(j_{*u}^- B(\zeta)), I_i(j_{*u}^- B(\tau))) = R(X, Y, Z, T) - \\
 & - \frac{tSc^2}{64n(n+2)^2} \{g(Y, J_1 Z)g(X, J_1 T) + g(Y, J_2 Z)g(X, J_2 T) - \\
 & - g(X, J_1 Z)g(Y, J_1 T) - g(X, J_2 Z)g(Y, J_2 T)\} + \\
 & + (\frac{Sc}{4n(n+2)} - \frac{2tSc^2}{64n(n+2)^2})(g(X, J_1 Y)g(Z, J_1 T) + g(X, J_2 Y)g(Z, J_2 T))
 \end{aligned}$$

for $i = 1, 2, \varepsilon_1 = -1, \varepsilon_2 = +1, u \in P(M), A, B, C, D \in m_3, \xi, \eta, \zeta, \tau \in \mathbb{R}^{4n}$ and $u(\xi) = X, u(\eta) = Y, u(\zeta) = Z, u(\tau) = T$. Now the conclusion follows from (20) and (22).

(iv) + (v) + (vi) Using (20) we obtain the following formulas for the Ricci tensors ρ^- of (Z^-, h_t) :

$$\begin{aligned}
 (23) \quad & \rho^-(j_{*u}^- A^*, j_{*u}^- B^*) = [\frac{tSc^2}{16(n+2)^2} + \frac{1}{nt}]h_t(j_{*u}^- A^*, j_{*u}^- B^*) \\
 & \rho^-(j_{*u}^- A^*, j_{*u}^- B(\xi)) = 0 \\
 & \rho^-(j_{*u}^- B(\xi), j_{*u}^- B(\eta)) = [\frac{Sc}{4n} - \frac{tSc^2}{16n(n+2)^2}]h_t(j_{*u}^- B(\xi), j_{*u}^- B(\eta))
 \end{aligned}$$

Similarly, from (15) and (20) we get *-Ricci tensors ρ_1^{*-} of (Z^-, I_1, h_t) , ρ_2^{*-} of (Z^-, I_2, h_t) :

$$\begin{aligned}
 (24) \quad & \rho_1^{*-}(j_{*u}^- A^*, j_{*u}^- B^*) = [-\frac{tSc^2}{16(n+2)^2} + \frac{Sc}{2(n+2)} + \frac{1}{nt}]h_t(j_{*u}^- A^*, j_{*u}^- B^*) \\
 & \rho_2^{*-}(j_{*u}^- A^*, j_{*u}^- B^*) = [\frac{tSc^2}{16(n+2)^2} - \frac{Sc}{2(n+2)} + \frac{1}{nt}]h_t(j_{*u}^- A^*, j_{*u}^- B^*) \\
 & \rho_1^{*-}(j_{*u}^- A^*, j_{*u}^- B(\xi)) = \rho_2^{*-}(j_{*u}^- A^*, j_{*u}^- B(\xi)) = 0 \\
 & \rho_1^{*-}(j_{*u}^- B(\xi), j_{*u}^- B(\eta)) = [\frac{Sc(n+1)}{4n(n+2)}]h_t(j_{*u}^- B(\xi), j_{*u}^- B(\eta)) \\
 & \rho_2^{*-}(j_{*u}^- B(\xi), j_{*u}^- B(\eta)) = [\frac{tSc^2}{16n(n+2)^2} + \frac{Sc(n-1)}{4n(n+2)}]h_t(j_{*u}^- B(\xi), j_{*u}^- B(\eta))
 \end{aligned}$$

Now (iv), (v) and (vi) follows from (23) and (24). \square

Theorem 3.2. Let (M, σ, g) be a $4n$ -dimensional paraquaternionic Kähler manifold and (Z^+, P_i, h_t) , $i = 1, 2$, the reflector spaces associated. Then:

- (i) The manifolds (Z^+, P_i, h_t) , $i = 1, 2$, belong always to APH2 and APH3 and are with para-Hermitian Ricci tensor and with para-Hermitian *-Ricci tensor;
- (ii) The manifolds (Z^+, P_1, h_t) belong to APH1 iff $Sc = 0$ or $Sc = -\frac{4(n+2)}{t}$;

- (iii) The manifolds (Z^+, P_2, h_t) belong to APH1 iff $Sc = 0$;
 (iv) The manifolds $(Z^+, P_i, h_t), i = 1, 2$ are Einstein iff

$$Sc = -\frac{4(n+2)}{t} \text{ or } Sc = -\frac{4(n+2)}{(n+1)t};$$

- (v) The manifolds (Z^+, P_1, h_t) are *-Einstein iff

$$Sc = -\frac{4(n+2)}{t} \text{ or } Sc = \frac{4(n+2)}{nt};$$

- (vi) The manifolds (Z^+, P_2, h_t) are *-Einstein iff

$$Sc = \frac{2(n+2)}{(n-1)t} [1 - 3n - \sqrt{9n^2 - 10n + 5}]$$

or

$$Sc = \frac{2(n+2)}{(n-1)t} [1 - 3n + \sqrt{9n^2 - 10n + 5}].$$

Proof. (i) This statement follows from (1), (9), (14), (16) and (21) after some long but straightforward computations.

(ii) + (iii) From (16) and (21) we obtain

$$\begin{aligned} & K^+(j_{*u}^+ A^*, j_{*u}^+ B^*, P_i(j_{*u}^+ C^*), P_i(j_{*u}^+ D^*)) = -\frac{t}{n}(J_1^0 A, B)(J_1^0 C, D) \\ & K^+(j_{*u}^+ A^*, j_{*u}^+ B^*, P_i(j_{*u}^+ C^*), P_i(j_{*u}^+ B(\xi))) = 0 \\ & K^+(j_{*u}^+ A^*, j_{*u}^+ B(\xi), P_i(j_{*u}^+ B^*), P_i(j_{*u}^+ B(\eta))) = \\ (25) & (\varepsilon_i \frac{t^2 Sc^2}{64n(n+2)^2} + \varepsilon_i \frac{tSc}{8n(n+2)})(A, B)g(X, Y) + \varepsilon_i \frac{t^2 Sc^2}{64n(n+2)^2}(J_1^0 A, B)g(J_1 X, Y) \\ & K^+(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta), P_i(j_{*u}^+ B(\zeta)), P_i(j_{*u}^+ A^*)) = 0 \\ & K^+(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta), P_i(j_{*u}^+ B(\zeta)), P_i(j_{*u}^+ B(\tau))) = -R(X, Y, Z, T) - \\ & -\frac{tSc^2}{64n(n+2)^2} \{g(Y, J_2 Z)g(X, J_2 T) - g(Y, J_3 Z)g(X, J_3 T) - \\ & -g(X, J_2 Z)g(Y, J_2 T) + g(X, J_3 Z)g(Y, J_3 T)\} + \\ & + (\frac{Sc}{4n(n+2)} + \frac{2tSc^2}{64n(n+2)^2})(-g(X, J_2 Y)g(Z, J_2 T) + g(X, J_3 Y)g(Z, J_3 T)) \end{aligned}$$

for $i = 1, 2, \varepsilon_1 = -1, \varepsilon_2 = +1, u \in P(M), A, B, C, D \in m_1, \xi, \eta, \zeta, \tau \in \mathbb{R}^{4n}$ and $u(\xi) = X, u(\eta) = Y, u(\zeta) = Z, u(\tau) = T$.

Now the conclusion follows from (21) and (25).

(iv) + (v) + (vi) Using (21) we obtain the following formulas for the Ricci tensors ρ^+ of (Z^+, h_t) :

$$\begin{aligned} & \rho^+(j_{*u}^+ A^*, j_{*u}^+ B^*) = [-\frac{tSc^2}{16(n+2)^2} - \frac{1}{nt}]h_t(j_{*u}^+ A^*, j_{*u}^+ B^*) \\ (26) \quad & \rho^+(j_{*u}^+ A^*, j_{*u}^+ B(\xi)) = 0 \\ & \rho^+(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta)) = [\frac{Sc}{4n} + \frac{tSc^2}{16n(n+2)^2}]h_t(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta)). \end{aligned}$$

Similarly, from (16) and (21) we get *-Ricci tensors ρ_1^{*+} of (Z^+, P_1, h_t) and ρ_2^{*+} of (Z^+, P_2, h_t) :

$$\begin{aligned}
\rho_1^{*+}(j_{*u}^+ A^*, j_{*u}^+ B^*) &= [-\frac{tSc^2}{16(n+2)^2} - \frac{Sc}{2(n+2)} + \frac{1}{nt}]h_t(j_{*u}^+ A^*, j_{*u}^+ B^*) \\
\rho_2^{*+}(j_{*u}^+ A^*, j_{*u}^+ B^*) &= [\frac{tSc^2}{16(n+2)^2} + \frac{Sc}{2(n+2)} + \frac{1}{nt}]h_t(j_{*u}^+ A^*, j_{*u}^+ B^*) \\
(27) \quad \rho_1^{*+}(j_{*u}^+ A^*, j_{*u}^+ B(\xi)) &= \rho_2^{*+}(j_{*u}^+ A^*, j_{*u}^+ B(\xi)) = 0 \\
\rho_1^{*+}(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta)) &= [-\frac{Sc(n+1)}{4n(n+2)}]h_t(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta)) \\
\rho_2^{*+}(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta)) &= [\frac{tSc^2}{16n(n+2)^2} - \frac{Sc(n-1)}{4n(n+2)}]h_t(j_{*u}^+ B(\xi), j_{*u}^+ B(\eta)).
\end{aligned}$$

Now (iv), (v) and (vi) follows from (26) and (27). \square

Remark 3.3. We note that the corresponding result of the Theorem 3.1 in quaternionic setting was obtained in [28] and the assertions stated at items (iv) in Theorems 3.1 and 3.2 were also proved, but using a different method, by Alekseevsky and Cortés (compare with [4, Corollary 6, page 126]).

4. MIXED 3-SASAKIAN STRUCTURES IN A $SO(2, 1)$ -PRINCIPAL BUNDLE OVER A PARAQUATERNIONIC KÄHLER MANIFOLD

Definition 4.1. Let M be a smooth manifold equipped with a triple (φ, ξ, η) , where φ is a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on M . If we have:

$$(28) \quad \varphi^2 = \tau(-I + \eta \otimes \xi), \quad \eta(\xi) = 1$$

then we say that:

- (i) (φ, ξ, η) is an *almost contact structure* on M , if $\tau = 1$ ([29]).
- (ii) (φ, ξ, η) is an *almost paracontact structure* on M , if $\tau = -1$ ([30]).

We remark that many authors also include in the above definition the conditions that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0,$$

although these are deducible from (28) (see [31]).

Definition 4.2. [13] A *mixed 3-structure* on a smooth manifold M is a triple of structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1, 2, 3\}$, which are almost paracontact structures for $\alpha = 1, 2$ and almost contact structure for $\alpha = 3$, satisfying the following conditions:

$$(29) \quad \eta_\alpha(\xi_\beta) = 0,$$

$$(30) \quad \varphi_\alpha(\xi_\beta) = \tau_\beta \xi_\gamma, \quad \varphi_\beta(\xi_\alpha) = -\tau_\alpha \xi_\gamma,$$

$$(31) \quad \eta_\alpha \circ \varphi_\beta = -\eta_\beta \circ \varphi_\alpha = \tau_\gamma \eta_\gamma,$$

$$(32) \quad \varphi_\alpha \varphi_\beta - \tau_\alpha \eta_\beta \otimes \xi_\alpha = -\varphi_\beta \varphi_\alpha + \tau_\beta \eta_\alpha \otimes \xi_\beta = \tau_\gamma \varphi_\gamma,$$

where (α, β, γ) is an even permutation of $(1, 2, 3)$ and $\tau_1 = \tau_2 = -\tau_3 = -1$.

Moreover, if a manifold M with a mixed 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}$ admits a semi-Riemannian metric g such that:

$$(33) \quad g(\varphi_\alpha X, \varphi_\alpha Y) = \tau_\alpha [g(X, Y) - \varepsilon_\alpha \eta_\alpha(X) \eta_\alpha(Y)],$$

for all $X, Y \in \Gamma(TM)$ and $\alpha = 1, 2, 3$, where $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$, then we say that M has a *metric mixed 3-structure* and g is called a *compatible metric*.

Remark 4.3. From (33) we obtain

$$(34) \quad \eta_\alpha(X) = \varepsilon_\alpha g(X, \xi_\alpha), \quad g(\varphi_\alpha X, Y) = -g(X, \varphi_\alpha Y)$$

for all $X, Y \in \Gamma(TM)$ and $\alpha = 1, 2, 3$.

Note that if $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ is a manifold with a metric mixed 3-structure then from (34) it follows

$$g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -g(\xi_3, \xi_3).$$

Hence the vector fields ξ_1 and ξ_2 are both either space-like or time-like and these force the causal character of the third vector field ξ_3 . We may therefore distinguish between *positive* and *negative metric mixed 3-structures*, according as ξ_1 and ξ_2 are both space-like, or both time-like vector fields. Because one can check that, at each point of M , there always exists a pseudo-orthonormal frame field given by $\{(E_i, \varphi_1 E_i, \varphi_2 E_i, \varphi_3 E_i)_{i=\overline{1,n}}, \xi_1, \xi_2, \xi_3\}$ we conclude that the dimension of the manifold is $4n + 3$ and the signature of g is $(2n + 1, 2n + 2)$, where we put first the minus signs, if the metric mixed 3-structure is positive (*i.e.* $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$), or the signature of g is $(2n + 2, 2n + 1)$, if the metric mixed 3-structure is negative (*i.e.* $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -1$).

Definition 4.4. [13] Let M be a manifold endowed with a (positive/negative) metric mixed 3-structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$. This structure is said to be:

- (i) a (positive/negative) *metric mixed 3-contact structure* if $d\eta_\alpha = \Phi_\alpha$, for each $\alpha \in \{1, 2, 3\}$, where Φ_α is the fundamental 2-form defined by $\Phi_\alpha(X, Y) := g(X, \varphi_\alpha Y)$.
- (ii) a (positive/negative) *mixed 3-K-contact structure* if $((\varphi_3, \xi_3, \eta_3), g)$ is a K-contact structure and $((\varphi_1, \xi_1, \eta_1), g), ((\varphi_2, \xi_2, \eta_2), g)$ are para-K-contact structures, *i.e.* $\nabla \xi_\alpha = -\varepsilon_\alpha \varphi_\alpha$, for each $\alpha \in \{1, 2, 3\}$, where ∇ is Levi-Civita connection of g .
- (iii) a (positive/negative) *mixed 3-Sasakian structure* if $(\varphi_3, \xi_3, \eta_3, g)$ is a Sasakian structure and $(\varphi_1, \xi_1, \eta_1, g), (\varphi_2, \xi_2, \eta_2, g)$ are para-Sasakian structures, *i.e.*

$$(35) \quad (\nabla_X \varphi_\alpha)Y = \tau_\alpha [g(X, Y)\xi_\alpha - \varepsilon_\alpha \eta_\alpha(Y)X]$$

for all $X, Y \in \Gamma(TM)$ and $\alpha \in \{1, 2, 3\}$.

Remark 4.5. Note that in fact mixed metric 3-contact, mixed 3-K-contact and mixed 3-Sasakian structures define the same class of manifolds, as proved in [13]. Moreover, like their Riemannian counterparts, mixed 3-Sasakian structures are Einstein, but now the scalar curvature can be either positive or negative:

Theorem 4.6. [13, 32] *Any $(4n + 3)$ -dimensional manifold endowed with a mixed 3-Sasakian structure is an Einstein space with Einstein constant $\lambda = (4n + 2)\varepsilon$, with $\varepsilon = \mp 1$, according as the metric mixed 3-structure is positive or negative, respectively.*

Remark 4.7. Several examples of manifolds endowed with mixed 3-Sasakian structures can be found in [33]. We note that the unit pseudo-sphere $S_{2n+2}^{4n+3} \subset \mathbb{R}_{2n+2}^{4n+4}$ is the canonical example of manifold with negative mixed 3-Sasakian structure, while the pseudo-hyperbolic space $H_{2n+1}^{4n+3} \subset \mathbb{R}_{2n+2}^{4n+4}$ can be endowed with a canonical positive mixed 3-Sasakian structure.

Let (M, σ, g) be a paraquaternionic Kähler manifold and let P be the bundle associated with (M, σ, g) . That is, P is the bundle whose transition functions and

structure group are the same as σ , but whose fibre is $\mathrm{SO}(2,1)$. The Lie algebra of $\mathrm{SO}(2,1)$ is $\mathfrak{so}(2,1)$ and we consider two of its bases: $\mathcal{B}^+ = \{e_1, e_2, e_3\}$ and $\mathcal{B}^- = \{-e_1, -e_2, -e_3\}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily verified that

$$(36) \quad [e_1, e_2] = 2e_3, \quad [e_2, e_3] = -2e_1, \quad [e_3, e_1] = -2e_2,$$

and

$$(37) \quad [-e_1, -e_2] = -2(-e_3), \quad [-e_2, -e_3] = 2(-e_1), \quad [-e_3, -e_1] = 2(-e_2).$$

On the other hand, their exp are

$$\begin{aligned} \exp(te_1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh 2t & \sinh 2t \\ 0 & \sinh 2t & \cosh 2t \end{pmatrix}, \\ \exp(te_2) &= \begin{pmatrix} \cosh 2t & 0 & \sinh 2t \\ 0 & 1 & 0 \\ \sinh 2t & 0 & \cosh 2t \end{pmatrix}, \\ \exp(te_3) &= \begin{pmatrix} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Then, with respect to the basis \mathcal{B}^+ , we have

$$\mathrm{ad}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \mathrm{ad}(e_2) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(e_3) = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and with respect to the basis \mathcal{B}^- , we have

$$\mathrm{ad}(-e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \quad \mathrm{ad}(-e_2) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(-e_3) = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For each neighborhood U of (M, σ, g) , we define a $\mathfrak{so}(2,1)$ -valued 1-form on U by

$$(38) \quad \omega_U = \sum_{i=1}^3 \omega_i e_i.$$

From (3) and (38) we obtain that on $U \cap U'$ we have

$$\omega_{U'}(X) = \mathrm{ad}(S_{UU'}^{-1}) \cdot \omega_U(X) + (S_{UU'})_*(X) \cdot S_{UU'}^{-1},$$

for all vector fields X on P , where $(S_{UU'})_*$ denotes the differential of the mapping $S_{UU'} : U \cap U' \rightarrow \mathrm{SO}(2,1)$. Therefore there exists a connection form ω on P such that

$$\psi^* \omega = \omega_U,$$

where ψ is a certain local cross-section of P over U .

The curvature form Ω defined by the connection ω is a $\mathfrak{so}(2, 1)$ -valued 2-form given by

$$\Omega(X, Y) = d\omega(X, Y) + \frac{1}{2}[\omega(X), \omega(Y)]$$

for all vector fields X, Y on P . Hence, using (36) we derive:

$$\psi^*\Omega = (d\omega_1 - \omega_2 \wedge \omega_3)e_1 + (d\omega_2 - \omega_3 \wedge \omega_1)e_2 + (d\omega_3 + \omega_1 \wedge \omega_2)e_3.$$

Theorem 4.8. *The principal bundle P associated with a paraquaternionic Kähler manifold (M, σ, g) of non-zero scalar curvature can be endowed with both a positive and a negative mixed 3-Sasakian structure.*

Proof. If $\omega_U = \sum_{i=1}^3 \omega_i e_i$ is the infinitesimal connection in P considered above, then we can define a semi-Riemannian metric \tilde{g} on P by

$$(39) \quad \tilde{g} = \nu \pi_* g + \sum_{i=1}^3 \varepsilon_i \omega_i \otimes \omega_i,$$

where ν is the reduced scalar curvature of M and $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, -1, 1)$ (Case I) or $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, -1)$ (Case II).

We remark that unlike quaternionic setting (see [17]), the signature of \tilde{g} does not depend on the sign of the scalar curvature. In fact, the signature is $(2n+2, 2n+1)$ in the Case I and $(2n+1, 2n+2)$ in the Case II.

Firstly, we consider the Case I. We define the 1-forms $\eta_\alpha = \omega_\alpha$ for $\alpha = 1, 2, 3$, and let ξ_1, ξ_2, ξ_3 be the fundamental vector fields corresponding to e_1, e_2, e_3 , respectively. Then from (36) we deduce

$$[\xi_1, \xi_2] = 2\xi_3, [\xi_2, \xi_3] = -2\xi_1, [\xi_3, \xi_1] = -2\xi_2$$

and

$$(40) \quad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3.$$

If $\tilde{\nabla}$ is the Levi-Civita connection of \tilde{g} , then we define

$$(41) \quad \varphi_\alpha = -\varepsilon_\alpha \tilde{\nabla} \xi_\alpha, \quad \alpha = 1, 2, 3.$$

Next we prove that $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}, \tilde{g})$ is a negative mixed 3-Sasakian structure on P .

If $p \in P$, then we denote by $T_p^V(P)$ the tangent space of a fibre at p and by $T_p^H(P)$ its orthogonal complemented space in $T_p P$. Using now (41) and Koszul formula, we derive

$$(42) \quad \varphi_\alpha(\xi_\alpha) = 0, \quad \varphi_\alpha(\xi_\beta) = \tau_\beta \xi_\gamma, \quad \varphi_\beta(\xi_\alpha) = -\tau_\alpha \xi_\gamma.$$

From (42) it follows that $T_p^V(P)$ and $T_p^H(P)$ are invariant under the action of φ_α , $\alpha = 1, 2, 3$. Therefore we have:

$$(43) \quad \varphi_\alpha = \varphi_\alpha^H + \tau_\beta \eta_\beta \otimes \xi_\gamma - \tau_\gamma \eta_\gamma \otimes \xi_\beta$$

for any even permutation (α, β, γ) of $(1, 2, 3)$, where φ_α^H denotes the restricted actions of φ_α on $T_p^H(P)$, for $\alpha = 1, 2, 3$.

Now, from the structure equations (6), we obtain for each neighborhood U in M and a local cross section $\psi : U \rightarrow P$ the following relation

$$(d\omega_\alpha + \tau_\alpha \omega_\beta \wedge \omega_\gamma)(\psi_* X, \psi_* Y) = \tau_\alpha \nu g(J_\alpha X, Y),$$

for any even permutation (α, β, γ) of $(1, 2, 3)$, where ψ_* denotes the differential of ψ . Since the curvature form is horizontal, we have

$$(\varphi_\alpha - \tau_\beta \eta_\beta \otimes \xi_\gamma + \tau_\gamma \eta_\gamma \otimes \xi_\beta)(\psi_* X) = (\psi_* J_\alpha X)^H,$$

so we derive

$$(44) \quad \varphi_\alpha(\psi_* X) = (\psi_* J_\alpha X)^H + \tau_\beta \eta_\beta(\psi_* X) \xi_\gamma - \tau_\gamma \eta_\gamma(\psi_* X) \xi_\beta.$$

Using now (2), (40) and (44) we obtain

$$\begin{aligned} \varphi_\alpha^2(\psi_* X) &= \varphi_\alpha[(\psi_* J_\alpha X)^H + \tau_\beta \eta_\beta(\psi_* X) \xi_\gamma - \tau_\gamma \eta_\gamma(\psi_* X) \xi_\beta] \\ &= (\psi_* J_\alpha^2 X)^H - \tau_\beta \tau_\gamma \eta_\gamma(\psi_* X) \xi_\gamma - \tau_\gamma \tau_\beta \eta_\beta(\psi_* X) \xi_\beta \\ &= -\tau_\alpha(\psi_* X)^H - \tau_\beta \tau_\gamma [\eta_\gamma(\psi_* X) \xi_\gamma + \eta_\beta(\psi_* X) \xi_\beta] \end{aligned}$$

and since $\tau_\alpha \tau_\beta \tau_\gamma = 1$, we deduce

$$(45) \quad \begin{aligned} \varphi_\alpha^2(\psi_* X) &= -\tau_\alpha[(\psi_* X)^H + \eta_\gamma(\psi_* X) \xi_\gamma + \eta_\beta(\psi_* X) \xi_\beta] \\ &= -\tau_\alpha[\psi_* X - \eta_\alpha(\psi_* X) \xi_\alpha]. \end{aligned}$$

From (40) and (45) we conclude that $(\varphi_1, \xi_1, \eta_1)$ and $(\varphi_2, \xi_2, \eta_2)$ are almost para-contact structures on P , while $(\varphi_3, \xi_3, \eta_3)$ is an almost contact structure on P . Now we are able to prove the compatibility conditions between these structures.

Using again (2), (40) and (44) we obtain

$$\begin{aligned} \varphi_\alpha \varphi_\beta(\psi_* X) &= \varphi_\alpha[(\psi_* J_\beta X)^H + \tau_\gamma \eta_\gamma(\psi_* X) \xi_\alpha - \tau_\alpha \eta_\alpha(\psi_* X) \xi_\gamma] \\ &= (\psi_* J_\alpha J_\beta X)^H + \tau_\alpha \tau_\gamma \eta_\alpha(\psi_* X) \xi_\beta \\ &= \tau_\gamma[(\psi_* J_\gamma X)^H + \tau_\alpha \eta_\alpha(\psi_* X) \xi_\beta] \\ &= \tau_\gamma[\varphi_\gamma(\psi_* X) + \tau_\beta \eta_\beta(\psi_* X) \xi_\alpha] \end{aligned}$$

and taking into account that $\tau_\alpha \tau_\beta \tau_\gamma = 1$, we deduce

$$\varphi_\alpha \varphi_\beta(\psi_* X) = \tau_\gamma \varphi_\gamma(\psi_* X) + \tau_\alpha(\psi_* X) \xi_\alpha.$$

Similarly we find

$$\varphi_\beta \varphi_\alpha(\psi_* X) = -\tau_\gamma \varphi_\gamma(\psi_* X) + \tau_\beta(\psi_* X) \xi_\beta.$$

and so the compatibility condition (32) is verified. Analogously we find that the compatibility condition (31) is checked. Moreover, from (40) and (42) we have that (29) and (30) are also true.

On the other hand, using (39) and (44) we derive

$$\begin{aligned} \tilde{g}(\varphi_\alpha \psi_* X, \varphi_\alpha \psi_* Y) &= \nu g(J_\alpha X, J_\alpha Y) + \varepsilon_\beta \eta_\gamma(\psi_* X) \eta_\gamma(\psi_* Y) + \varepsilon_\gamma \eta_\beta(\psi_* X) \eta_\beta(\psi_* Y) \\ &= \tau_\alpha [\nu g(X, Y) + \varepsilon_\beta \eta_\beta(\psi_* X) \eta_\beta(\psi_* Y) + \varepsilon_\gamma \eta_\gamma(\psi_* X) \eta_\gamma(\psi_* Y)] \\ &= \tau_\alpha [\tilde{g}(\psi_* X, \psi_* Y) - \varepsilon_\alpha \eta_\alpha(\psi_* X) \eta_\alpha(\psi_* Y)]. \end{aligned}$$

Hence the metric \tilde{g} is compatible with $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, for $\alpha = 1, 2, 3$.

Therefore, taking account of (41), we deduce that $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \tilde{g})$ is a negative mixed 3-K-contact structure in P , and so, via Remark 4.5, is a negative mixed 3-Sasakian structure in P .

In the Case II, we consider ξ'_1, ξ'_2, ξ'_3 to be the fundamental vector fields corresponding to $-e_1, -e_2, -e_3$, respectively, and we define

$$\eta'_\alpha = -\omega_\alpha, \quad \varphi'_\alpha = -\varepsilon_\alpha \tilde{\nabla} \xi'_\alpha,$$

for $\alpha = 1, 2, 3$. Then, it can be proved similarly as in Case I that $((\varphi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=\overline{1,3}}, \tilde{g})$ is a positive mixed 3-Sasakian structure in P . \square

Remark 4.9. We note that in [34], the authors investigated the possible projectability onto paraquaternionic structures of metric mixed 3-contact structures defined on the total space of a G -principal bundle. In fact they proved that if $P(M, G, \pi)$ is a G -principal bundle, where the total space P is endowed with a metric mixed 3-contact structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$, G is a Lie group acting on the right on P by isometries having Lie algebra isomorphic to $\mathfrak{so}(2, 1)$ and such that the vector fields ξ_1, ξ_2, ξ_3 are fundamental vector fields, then the metric mixed 3-contact structure projects via π onto a paraquaternionic Kähler structure (σ, h) on the base space M and $\pi : (P, g) \rightarrow (M, h)$ is a pseudo-Riemannian submersion. Moreover, if the dimension of P is $(4n + 3)$, then one has

$$(46) \quad \rho^M = \varepsilon 4(n + 2)h$$

where ρ^M denotes the Ricci curvature tensor of M and $\varepsilon = \mp 1$, according as the mixed 3-structure is positive or negative.

Let now X, Y and U, V be basic and vertical vector fields throughout. The metric g splits up as

$$g(E, F) = g(X + U, Y + V) = g(X, Y) + g(U, V),$$

where $E = X + U, F = Y + V$ are vector fields on P . Then the canonical variation of the metric g for $t > 0$, is a pseudo-Riemannian metric g_t on P defined by (see [35])

$$(47) \quad g_t(X + U, Y + V) = g(X, Y) + tg(U, V).$$

Similarly as in the Riemannian case [36], it can be proved that the curvature tensor R_t of the canonical variation g_t satisfies

$$(48) \quad \begin{aligned} R_t(E, F, G, H) &= tR(E, F, G, H) + (1 - t)R^N(X_*, Y_*, Z_*, Z'_*) + \\ &\quad + t(1 - t)(g(S_E H, S_F G) - g(S_E G, S_F H)) \end{aligned}$$

where E, F, G, H are vector fields on P , X_*, Y_*, Z_*, Z'_* are the corresponding vector fields on M of the basic vector fields X, Y, Z, Z' and $S_E F = T_U V + A_X V + A_Y U$, $E = X + U, F = Y + V, G = Z + W, H = Z' + W'$, T and A being the O'Neill tensors of the submersion (see [37]). In this setting, we can state now the following

Theorem 4.10. *If $\pi : (P, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g) \rightarrow (M, \sigma, h)$ is a pseudo-Riemannian submersion as in the previous remark, then there exist a unique positive $t \neq 1$ such that g_t is also Einstein.*

Proof. We denote by R the curvature tensor of manifold P and by ρ, ρ^M the Ricci curvature tensor of manifold P , respectively M . Using (34) and (35) we obtain that

$$(49) \quad R(E, F)\xi_\alpha = \tau_\alpha(\eta_\alpha(F)E - \eta_\alpha(E)F), \quad \alpha = 1, 2, 3,$$

where E, F are vector fields on P .

Now using (47), (48), (49) and the definitions of mixed 3-Sasakian and paraquaternionic Kähler structures we obtain that the components of the Ricci tensor ρ_t of the canonical variation g_t are given by

$$\begin{aligned} \rho_t(U, V) &= t\rho(U, V) + \{-4n\varepsilon t^2 + (4n\varepsilon + 2\varepsilon)t - 2\varepsilon\}g(U, V), \\ \rho_t(U, X) &= t\rho(U, X), \\ \rho_t(X, Y) &= t\rho(X, Y) + (1 - t)\rho^M(X_*, Y_*), \end{aligned}$$

where X, Y and U, V are basic and vertical vector fields on P , X_*, Y_* are the vector fields on M corresponding to the basic vector fields X, Y , $\varepsilon = \mp 1$ according as the metric mixed 3-structure is positive or negative, respectively.

From Theorem 2.1, Theorem 4.6 and (46) we derive

$$\begin{aligned}
 \rho_t(U, V) &= \{-4n\varepsilon t^2 + (8n\varepsilon + 4\varepsilon)t - 2\varepsilon\}g(U, V), \\
 \rho_t(U, X) &= 0, \\
 \rho_t(X, Y) &= \{-6\varepsilon t + 4n\varepsilon + 8\varepsilon\}g(X, Y),
 \end{aligned}
 \tag{50}$$

Now, from (50) we deduce easily that g_t is Einstein if and only if $t = 1$ or $t = \frac{2n+5}{2n}$ and the conclusion follows. \square

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